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# THE GENERALIZED POLARIZATION SCATTERING MATRIX

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# THE GENERALIZED POLARIZATION SCATTERING MATRIX

## 1. INTRODUCTION

The conventional definition of the monostatic monocromatic polarization matrix is first extended to the bistatic case, then to the short pulse case, and finally to the bistatic short pulse case. The transformations and convolutions involved are discussed in some detail.

The method of determining the Least Square Best Estimate of the Generalized Polarization matrix from a set of measurements is then developed.

It is shown that the Faraday rotation angles introduced by a magneto ionic medium intervening the radar and the target are determinable from measured short pulse monostatic polarization matrix data.

It is then shown that the Least Square Best Estimate of the orientation angle of a symmetric target is also determinable from Faraday rotation contaminated short pulse monostatic polarization matrix data.

## 2. THE MONOSTATIC MONOCHROMATIC POLARIZATION SCATTERING MATRIX

The monostatic monochromatic polarization scattering matrix $\rho_{\mu\gamma}$ is defined by the equation[1] :

$$\boxed{S_\mu = \rho_{\mu\gamma} I_\gamma} \tag{1}$$

where $\overline{I}$ and $\overline{S}$ are the range normalized incident and scattered electric (or magnetic; depending on the convention adopted) far-fields respectively, the phases of which are referred to a coordinate system in which the scatterer is described (See Figure 1, p. 3); i. e.,

$$\overline{E}_i(\overline{x}) = I \; e^{i\overline{k}_i \cdot \overline{x}} \tag{2}$$

$$\overline{E}_s(\overline{x}) = \frac{1}{\sqrt{4\pi x^2}} \, \overline{S} \; e^{i\overline{k}_s \cdot \overline{x}} \tag{3}$$

where $\overline{E}_i(\overline{x})$ and $\overline{E}_s(\overline{x})$ are the incident and scattered electric fields at the point $\overline{x}$ respectively, and $\overline{k}_i$ and $\overline{k}_s$ are the propagation vectors of the incident and scattered fields respectively.

The rationale of the range normalization of the scattered field (see equation 3) is to insure consistency within the power cross-section definition[2] :

$$\sigma = 4\pi x^2 \; \frac{|\overline{E}_s|^2}{|\overline{E}_i|^2} \tag{4}$$

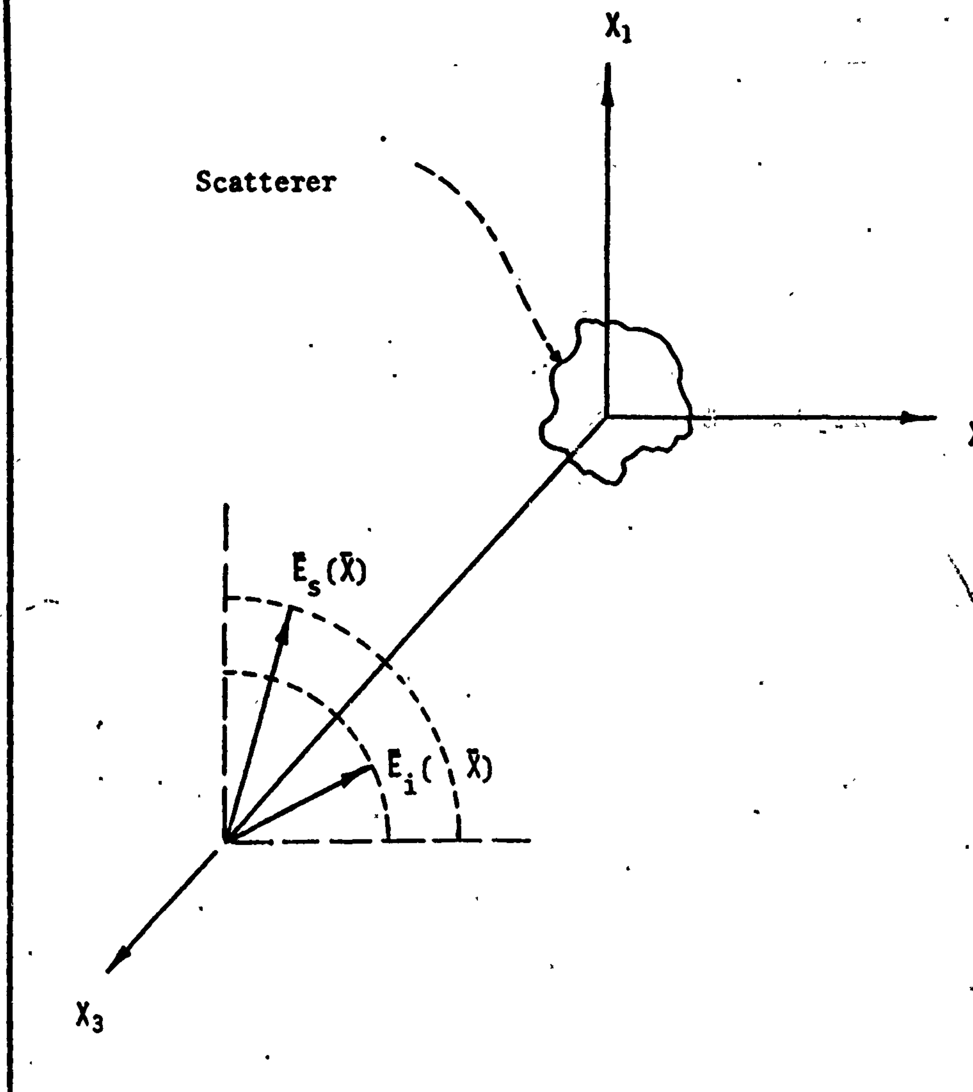


Figure 1

The Target Coordinate System

and the relationship:

$$\sigma = \rho \rho^{*} \tag{5}$$

where $\rho$ is the Eigen Value of the polarization matrix for the special case of certain geometries (to be discussed later) for which this polarization matrix is fully degenerate; i.e.:

$$\rho_{\mu\gamma} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \tag{6}$$

For this monostatic case, by the very definition of the monostatic case as $\overline{k}_s = \overline{k}_i$ (See Figure 1, p3):

$$\overline{k}_s = \frac{\omega}{c} \hat{k} \tag{7}$$

$$\overline{k}_i = - \frac{\omega}{c} \hat{k} \tag{8}$$

where $\hat{k}$ is the unit vector specifying the viewing direction, the boresight vector of the radar, and the aspect angles (of the radar relative to the target) for this monostatic case (See Figure 1, p. 3).

It should be noted that the matrix $\rho_{\mu\gamma}$ is not invariant to the choice of coordinate systems.

It should further be noted that for this monostatic case, by virtue of the transversality of the far-fields, the matrix $\rho_{\mu\gamma}$ can be fully described as a second rank matrix if $x_3$ is chosen as colinear with $\overline{k}_s$ (see Figure 1, p. 3); i.e., since $I_3=S_3=0$ for this coordinate system, it follows that $\rho_{3\mu} = \rho_{\mu 3} = 0$.

## 3. THE BISTATIC GENERALIZATION

For the monostatic case equation 1, by virtue of equations 7 and 8, can clearly be written as:

$$S_\mu(\hat{k}) = \rho_{\mu\gamma}(\hat{k})\, I_\gamma(\hat{k}) \tag{9}$$

For the bistatic case, for which $\bar{k}_i \neq -\bar{k}_s$, the natural generalization of the above defining equation 9 is clearly (see Figure 2, p. 6):

$$S_\mu(\hat{k}_s) = \rho_{\mu\gamma}(\hat{k}_s, \hat{k}_i)\, I_\gamma(\hat{k}_i) \tag{10}$$

Except that it is now no longer possible to find a single coordinate system such that the rank of the matrix $\rho_{\mu\gamma}(\hat{k}_s, \hat{k}_i)$ reduces to two for all $\hat{k}_i \neq -\hat{k}_s$. It is, however, possible to find two coordinate systems such that the matrix $\rho_{\mu\gamma}(\hat{k}_s, \hat{k}_i)$, expressed in these mixed coordinate systems, reduces to a matrix of rank two; i.e., any two coordinate systems $\bar{x}^{(i)}$ and $\bar{x}^{(s)}$ chosen such that $x_3^{(i)}$ and $x_3^{(s)}$ are colinear with $\hat{k}_i$ and $\hat{k}_s$ respectively. The details of the transformation of equation 10 into such a mixed coordinate system will be presented next.

Let $T_{\mu\gamma}^{(i)}$ and $T_{\mu\gamma}^{(s)}$ be the coordinate transformation matrices from the target coordinate system $\bar{x}$ (see Figure 1, p. 3) to the coordinate systems $\bar{x}^{(i)}$ and $\bar{x}^{(s)}$ respectively (see Figure 2, p. 6); i.e.:

$$I_\mu^{(i)} = T_{\mu\gamma}^{(i)}\, I_\gamma \tag{11}$$

$$S_\mu^{(s)} = T_{\mu\gamma}^{(s)}\, S_\gamma \tag{12}$$

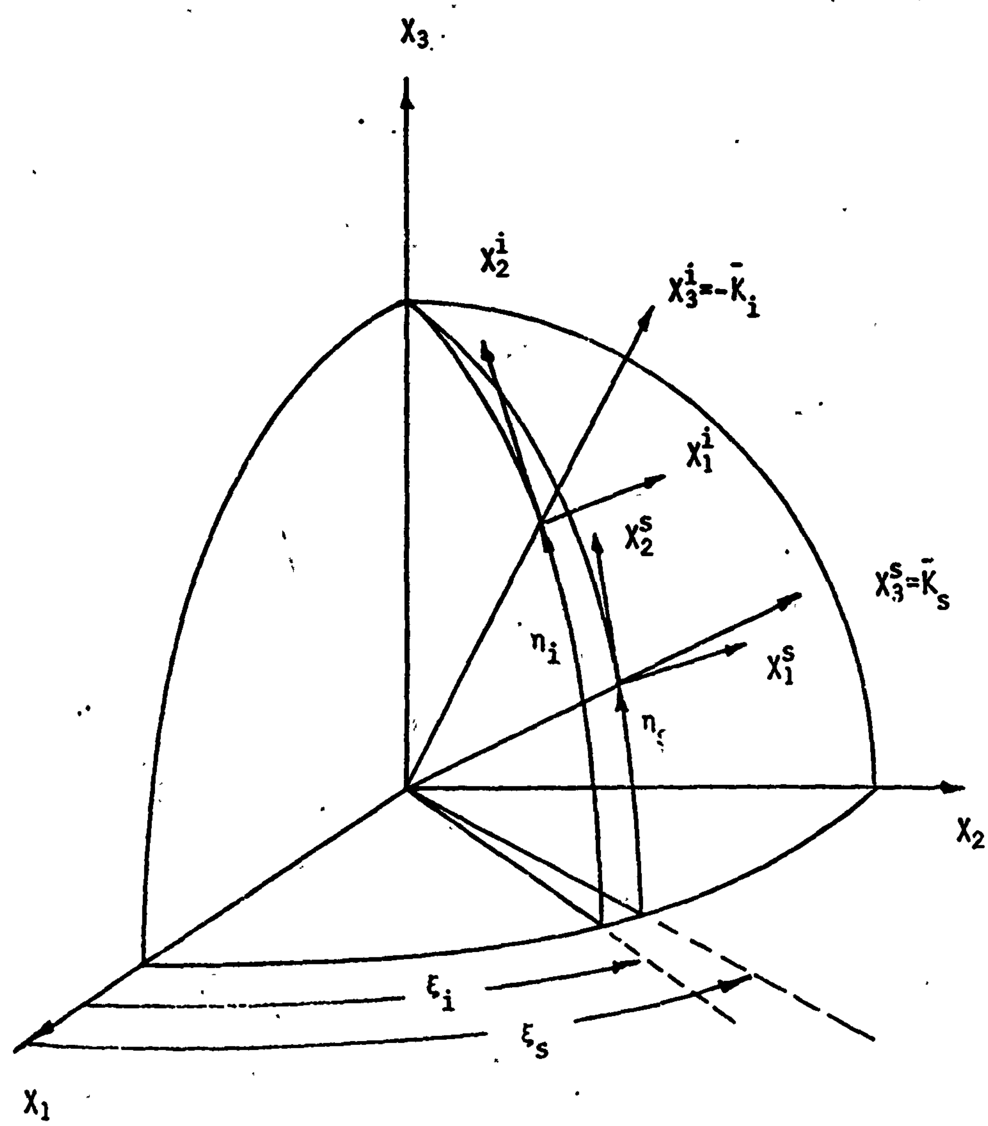


Figure 2

The Mixed Coordinate System

where (see Figure 2, p. 6):

$$T_{\mu\gamma}^{(n)} = \begin{pmatrix} -\sin\xi_n & \cos\xi_n & 0 \\ -\cos\xi_n \sin\eta_n & -\sin\xi_n \sin\eta_n & \cos\eta_n \\ \cos\xi_n \cos\eta_n & \sin\xi_n \cos\eta_n & \sin\eta_n \end{pmatrix}, \quad n = i, s \qquad (13)$$

It thus follows from equation 10 that (since $T^{-1} = \tilde{T}$ for a real unitary transformation):

$$\tilde{T}^{(s)} S^{(s)} = \rho\, \tilde{T}^{(i)} I^{(i)} \qquad (14)$$

$$S^{(s)} = T^{(s)} \rho\, \tilde{T}^{(i)} I^{(i)} \qquad (15)$$

The polarization matrix in the mixed coordinate system, say $\rho'$, is thus:

$$\rho' = T^{(s)} \rho\, \tilde{T}^{(i)} \qquad (16)$$

where, by virtue of the transversality of the incident and scattered far-fields, the matrix $\rho'$ is a second rank matrix; i.e.:

$$\boxed{\rho'_{\mu\gamma} = T^{(s)}_{\mu\alpha}\, T^{(i)}_{\gamma\beta}\, \rho_{\alpha\beta}} \qquad (17)$$

$$\mu, \gamma = 1, 2$$
$$\alpha, \beta = 1, 2, 3$$

Relative to the $x_3$-axis, the components 1 and 2 of the primed second rank matrix are clearly the TRANSVERSE and LONGITUDINAL components respectively (see Figure 2, p. 6); e.g., if $x_3$ is the axis of symmetry of an axially symmetric scatterer.

## 4. THE SHORT PULSE GENERALIZATION

For the monochromatic case, equation 1, by virtue of equations 7 and 8, can clearly be written as:

$$\boxed{S_\mu(\omega) = \rho_{\mu\gamma}(\omega)\ I_\gamma(\omega)} \tag{18}$$

For the time domain, equation 18 thus yields:

$$\boxed{S_\mu(t) = \rho_{\mu\gamma}(t) * I_\gamma(t)} \tag{19}$$

where:

$$\rho_{\mu\gamma}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \rho_{\mu\gamma}(\omega)\, d\omega \tag{20}$$

$$S_\mu(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} S_\mu(\omega)\, d\omega \tag{21}$$

$$I_\mu(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} I_\mu(\omega)\, d\omega \tag{22}$$

In the time domain, the polarization matrix equation (equation 19) is thus a matrix-convolution equation; i.e.,

$$S_\mu(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho_{\mu\gamma}(t')\, I_\gamma(t-t')dt' \tag{23}$$

For finite transmitted power, i.e.:

$$\int_{-\infty}^{\infty} I_\gamma(t)\, I_\gamma^*(t)\, dt < \infty \tag{24}$$

it follows that:

$$\int_{-\infty}^{\infty} \left| \rho_{\mu\gamma}(t') I_{\gamma}(t-t') \right| dt' < \infty \tag{25}$$

It thus follows that the integration and implied summation in equation 23 can be interchanged; i.e., written out:

$$\boxed{S_{\mu}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\gamma=1}^{2} \int_{-\infty}^{\infty} \rho_{\mu\gamma}(t') I_{\gamma}(t-t') \, dt'} \tag{26}$$

## 5. THE BISTATIC SHORT PULSE GENERALIZATION

For the monostatic monochromatic case equation 1, by virtue of equations 7 and 8, can clearly be written as:

$$S_{\mu}(\omega, \hat{k}) = \rho_{\mu\gamma}(\omega, \hat{k}) I_{\gamma}(\omega, \hat{k}) \tag{27}$$

Thus, for the bistatic monochromatic case, by virtue of equation 10:

$$\boxed{S_{\mu}(\omega, \hat{k}) = \rho_{\mu\gamma}(\omega, \hat{k}_s, \hat{k}_i) I_{\gamma}(\omega, \hat{k}_i)} \tag{28}$$

Thus, in the time domain, by virtue of the results of equation 26:

$$S_{\mu}(t, \hat{k}_s) = \frac{1}{\sqrt{2\pi}} \sum_{\gamma=1}^{3} \int_{-\infty}^{\infty} \rho_{\mu\gamma}(t', \hat{k}_s, \hat{k}_i) I(t-t', \hat{k}_i) \, dt' \tag{29}$$

In the mixed coordinate system discussed in Section 3, equation 29 yeilds:

$$S_{\mu}^{(s)}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\gamma=1}^{2} \int_{-\infty}^{\infty} \rho_{\mu\gamma}(t') \, I_{\gamma}^{(i)}(t-t') \, dt' \tag{30}$$

where by equations 11, 12, and 17; and equations 20, 21, and 22:

$$I_{\mu}^{(i)}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\gamma=1}^{3} \tau_{\mu\gamma}(\hat{k}_i) \int_{-\infty}^{\infty} e^{-i\omega t} I_{\gamma}(\omega, \hat{k}_i) \, d\omega \tag{31}$$

$$S_{\mu}^{(s)}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\gamma=1}^{3} \tau_{\mu\gamma}(\hat{k}_s) \int_{-\infty}^{\infty} e^{-i\omega t} S_{\gamma}(\omega, \hat{k}_s) \, d\omega \tag{32}$$

$$\boxed{\rho'_{\mu\gamma}(t) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \tau_{\mu\alpha}(\hat{k}_s) \, \tau_{\gamma\beta}(\hat{k}_i) \int_{-\infty}^{\infty} e^{-i\omega t} \rho_{\alpha\beta}(\omega, \hat{k}_s, \hat{k}_i) \, d\omega} \tag{33}$$

where the logical notation:

$$\tau_{\mu\gamma}(k_n) \equiv \tau_{\mu\gamma}^{(n)} \; ; \qquad n = i, s \tag{34}$$

has been adopted.

## 6. THE MEASUREMENT OF THE POLARIZATION MATRIX

A single measurement with a monochromatic monostatic radar possessing polarization diversity capabilities (i.e., a radar capable of transmitting an arbitrary incident polarization and measuring the scattered polarization associated with this transmission) clearly yields knowledge of the pair of vectors $\overline{I}$ and $\overline{S}$ (see equation 1, p. 2). For a set of such measurements, a set of pairs of vectors, say, $\overline{I}^{(n)}$ and $\overline{S}^{(n)}$ is clearly yielded. The Least Square Best Estimate of the polarization matrix (say $\breve{\rho}$) is thus that matrix $\breve{\rho}$ which best satisfies equation 1, p. 2; i.e., the Least Square Best Estimate of $\rho$ by the equation:

$$S_{\mu}^{(n)} = \rho_{\mu\gamma} I_{\gamma}^{(n)} \tag{35}$$

Defining the matrices:

$$S_{\mu n} \equiv S_{\mu}^{(n)} \tag{36}$$

$$I_{\gamma n} \equiv I_{\gamma}^{(n)} \tag{37}$$

this yields for equation 35:

$$S_{\mu n} = \rho_{\mu\gamma} I_{\gamma n} \tag{38}$$

or, in pure matrix notation:

$$S = \rho I \tag{39}$$

If the matrices, I, S, and $\rho$ were real, when the Least Square Best Estimate of $\rho$ would have been given by[3]:

$$\breve{\rho} = S\tilde{I}\,(I\,\tilde{I})^{-1} \tag{40}$$

It can readily be shown[4] that for complex matrices I, S, and $\rho$, the Least Square Best Estimate of $\rho$ is:

$$\boxed{\breve{\rho} = S I^{\dagger} (I I^{\dagger})^{-1}} \tag{41}$$

where the error associated with the Least Square Best Estimate is calculable by the conventional method[5].

For the Bistatic Short Pulse Case (see equation 28, p. 9), the defining equations 36 and 37 need thus merely be extended to:

$$S_{\mu n}(\omega, \hat{k}_s) = S_{\mu}^{(n)}(\omega, \hat{k}_s) \tag{42}$$

$$I_{\gamma n}(\omega, \hat{k}_s) = I_{\gamma}^{(n)}(\omega, \hat{k}_s) \tag{43}$$

The additional constraining condition that the polarization matrix is true-symmetric[6] has not been imposed on the Least Square Best Estimate of the polarization matrix for reasons that will become evident in the next Section 7.

## 7. THE FARADAY ROTATION ANGLE

If a mangeto ionic medium (such as the ionosphere in the magnetic field of the earth) is intervening between the radar and the target, then the measured monostatic monochromatic polarization matrix is not true-symmetric because of the Faraday rotation effects introduced by such a medium[7].

Given such a Faraday rotation contaminated monostatic monochromatic polarization matrix, the Faraday rotation angle $\theta$ can be determined[8]; i.e.:

$$\boxed{\tan 2\theta = \frac{\rho_{12} - \rho_{21}}{\rho_{11} + \rho_{22}}} \tag{44}$$

and the uncontaminated monostatic monochromatic polarization matrix, say $\rho^{\bullet}$, (namely that matrix that should have been measured had there been no intervening magneto ionic medium) can also be determined[9]; i.e.:

$$\rho^{\bullet} = \Theta \rho \Theta \tag{45}$$

where:

$$\Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{46}$$

The rationale for not seeking a true-symmetric Least Square Best Estimate of the polarization matrix (see Section 6) is thus evident; namely, the asymmetric components of the Least Square Best Estimate of the polarization matrix are utilized in equation 44.

Since the preceding equations 44, 45, and 46 are in the frequency domain, it follows by an argument similar to the argument of Section 4 that for the short pulse-monostatic polarization matrix:

$$\theta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\theta(\omega)d\omega$$

where $\theta(\omega)$ is given by equation 44, and

$$\rho^{\bullet}(t) = \Theta(t) * \rho(t) * \Theta(t) \tag{48}$$

where:

$$\Theta(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \qquad (49)$$

## 8. THE ORIENTATION ANGLE OF A SYMMETRIC TARGET

It has previously been shown[10] that the orientation angle $\alpha$ of a symmetric target is related to the Faraday rotation contaminated monostatic monochromatic polarization matrix $\rho$ by:

$$\boxed{\tan 2\alpha = \frac{\rho_{12} + \rho_{21}}{\rho_{11} - \rho_{22}}} \qquad (50)$$

Since the above equation 50 is in the frequency domain, it follows by an argument similar to the argument of Section 4 that the above equation 50 must be written as:

$$\tan 2\alpha(\omega) = \frac{\rho_{12}(\omega) + \rho_{21}(\omega)}{\rho_{11}(\omega) - \rho_{22}(\omega)} \qquad (51)$$

The orientation angle $\alpha$ is clearly, however, not a function of the frequency, since it is a purely geometrical quantity.

It thus immediately follows that the Least Square Best Estimate of the orientation angle $\alpha$, say $\check{\alpha}$, is given for the short pulse monostatic polarization matrix case by:

$$\check{\alpha} = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \alpha(\omega)\, d\omega \qquad (52)$$

where:

$$\alpha(\omega) = \frac{1}{2} \tan^{-1} \left[ \frac{\check{\rho}_{12}(\omega) + \check{\rho}_{21}(\omega)}{\check{\rho}_{11}(\omega) + \check{\rho}_{22}(\omega)} \right] \tag{53}$$

where $\check{\rho}$ is given by equation 41 of Section 6.

(U) It should be noted in conclusion that the determination of the Least Square Best Estimate of the orientation angle is accomplished directly, without first determining the Faraday rotation angle and then removing its effects.

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